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DOI:

[10.1016/j.physletb.2016.06.011](https://doi.org/10.1016/j.physletb.2016.06.011)

*Document Version*

Publisher's PDF, also known as Version of record

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*Citation for published version (APA):*

Tumanov, A., & West, P. (2016). E11 must be a symmetry of strings and branes. *Physics Letters, Section B: Nuclear, Elementary Particle and High-Energy Physics*, 759, 663-671.  
<https://doi.org/10.1016/j.physletb.2016.06.011>

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# $E_{11}$ must be a symmetry of strings and branes



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## ARTICLE INFO

### Article history:

Received 4 February 2016

Received in revised form 5 April 2016

Accepted 6 June 2016

Available online 8 June 2016

Editor: N. Lambert

## ABSTRACT

We construct the dynamical equations, at low levels, that are contained in the non-linear realisation of the semi-direct product of  $E_{11}$  and its vector representation in five and eleven dimensions. Restricting these equations to contain only the usual fields of supergravity and the generalised space-time to be the usual space-time we find the equations of motion of the five and eleven dimensional maximal supergravity theories. Since this non-linear realisation contains effects that are beyond the supergravity approximation and are thought to be present in an underlying theory we conclude that the low energy effective action of string and branes should possess an  $E_{11}$  symmetry.

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## 1. Introduction

It has been conjectured that the low energy effective action for strings and branes is the non-linear realisation of the semi-direct product of  $E_{11}$  and its vector ( $l_1$ ) representation, denoted  $E_{11} \otimes_s l_1$  [1,2]. This theory has an infinite number of fields, associated with  $E_{11}$ , which live on a generalised spacetime, associated with the vector representation  $l_1$ . The fields obey equations of motion that are a consequence of the non-linear realisation.

For many years, it was believed that strings by themselves could constitute a complete consistent theory whose precise form was yet to be found. However, while the perturbative quantum properties of string theory are very well understood, the situation is much less clear for their non-perturbative properties. Although supergravity theories in ten dimensions were not believed to be consistent theories of quantum gravity they did provide a low energy effective action for strings which included their non-perturbative effects. It became clear from these supergravity theories that the underlying theory must contain branes as well as strings. Unfortunately, it is not understood how to quantise a single brane and not much is known about how branes scatter. Thus it became clear that very little is known about the underlying consistent quantum theory that contains strings and branes.

For simplicity we will now restrict our discussion to maximally supersymmetric theories. Although, as we have just indicated, supergravity theories have provided a very useful starting point for discussions of an underlying theory there are quite a number of

maximal supergravity theories. These are one in eleven dimensions, two in ten dimensions and one in each dimension below ten. In addition there also a substantial number of so called gauged supergravity theories which are obtained by adding a cosmological constant to a massless maximal supergravity theory in a given dimension and then seeing what theories are possible whilst preserving all the supersymmetries [3].

Although there are quite a few maximal supergravity theories, and so string theories, some of them are related to each other by duality transformation; such as the T duality between the IIA and IIB supergravity theories in ten dimensions and the relation between the IIA supergravity theory and the eleven supergravity theory. Such relations were summarised in what became known as M theory, but M theory is not so much a unified theory but more a set of observations between different theories. Furthermore, M theory does not include relations between the different gauge supergravities, or relate them to the massless supergravities in a systematic way.

The situation with  $E_{11}$  is different. The different theories emerge by decomposing  $E_{11}$  into different subgroups, in particular, the theory in  $D$  dimensions arises by considering the decomposition into  $GL(D) \otimes E_{11-d}$ , except in ten dimensions where there are two possible decompositions corresponding to the IIA and IIB theories [4–7]. The fields one finds at low levels in these different decompositions are just the bosonic fields of the corresponding supergravity theories. However, at the next level one finds next to spacefilling forms (fields with  $D - 1$  totally anti-symmetric indices); these lead to all the different gauged supergravities [6, 8]. As a result all the maximal supergravity theories, including the gauged supergravities, should be contained in a single theory, namely the non-linear realisation of  $E_{11} \otimes l_1$ . Some of the fields

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that arise at higher levels in  $E_{11}$  are known to be needed. One example is the ten forms predicted [9] in ten dimensions for the IIA and IIB theories which were confirmed to be present by showing that they are the unique such forms that are admitted by the supersymmetry algebras with which these theories are usually formulated [10].

There is also very good evidence that all the brane charges are contained in the vector representation [2,11–14]. However, the situation is less clear for the physical relevance of the infinite number of coordinates beyond those of the usual spacetime that arise in the  $E_{11} \otimes_s l_1$  non-linear realisation. Nonetheless, all gauged maximal supergravities in five dimensions were constructed by taking the fields to depend on some of the higher level generalised coordinates [7].

Partial results on the dynamics encoded in the  $E_{11} \otimes_s l_1$  non-linear realisation can be found in many of the early  $E_{11}$  papers. However, these papers often used only the usual coordinates of spacetime and also only the Lorentz part of the  $I_c(E_{11})$  local symmetry, as a result the full power of the symmetries of the non-linear realisation was not exploited. A more systematic approach was used to constructing the equations of motion of the  $E_{11} \otimes_s l_1$  non-linear realisation in eleven [15] and four [16] dimensions by including both the higher level generalised coordinates and local symmetries in  $I_c(E_{11})$ . The equations of motion of the form fields were derived, but these papers did not complete the derivation of the gravity equations and as a result the complete contact with the supergravity theories was not made. A more detailed account of previous papers which contained partial derivations of the equations of motion of the non-linear realisation  $E_{11} \otimes_s l_1$  can be found in [17].

In this paper we will rectify the just mentioned defect. We will, as before, derive the equations of motion for the form and scalar fields, this time in five dimensions, but we will also find the gravity equation at low level. The result is that if, at the end of the derivation, we systematically truncate to keep only the usual fields of supergravity and derivatives with respect to the usual coordinates of spacetime then the equations of motion are those of five dimensional maximal supergravity. In doing this we will choose the values of two constants which are not determined as a result of the limited extent to which the calculation has been carried out in this paper. These constants will, however, be fixed by taking subsequent variations of the equations and this will be given in a future publication.

In section four we will also evaluate the  $E_{11} \otimes_s l_1$  non-linear realisation in eleven dimensions. When restricted in the same way we find the equations of motion of eleven dimensional supergravity. However, in this case we will vary the gravity equation under the symmetries of the non-linear realisation, at the linearised level, to find the three form equation of motion. In so doing so we will fix the remaining undetermined constant. Thus we precisely recover the bosonic sector of eleven dimensional supergravity. As a result we have in effect shown the  $E_{11}$  conjecture [1,2]; namely that the  $E_{11} \otimes_s l_1$  non-linear realisation provides a unified low energy effective theory of strings and branes.

We begin by very briefly recalling the main features of the non-linear realisation of  $E_{11} \otimes_s l_1$  which is constructed from the group element  $g \in E_{11} \otimes_s l_1$  that can be written as

$$g = g_l g_E \quad (1.1)$$

In this equation  $g_E$  is a group element of  $E_{11}$  and so can be written in the form  $g_E = e^{A_{\underline{\alpha}} R^{\underline{\alpha}}}$  where the  $R^{\underline{\alpha}}$  are the generators of  $E_{11}$  and  $A_{\underline{\alpha}}$  are the fields in the non-realisation. The group element  $g_l$  is formed from the generators of the vector ( $l_1$ ) representation and so has the form  $e^{z^A L_A}$  where  $z^A$  are the coordinates of the generalised space-time. The fields  $A_{\underline{\alpha}}$  depend on the coordinates  $z^A$ .

The  $E_{11} \otimes_s l_1$  algebra and the explicit form of these group elements can be found in earlier papers on  $E_{11}$ , for example in dimensions eleven [15], five [7] and four [16]. The non-linear realisation is, by definition, invariant under the transformations

$$g \rightarrow g_0 g, \quad g_0 \in E_{11} \otimes_s l_1, \quad \text{as well as} \quad g \rightarrow g h, \quad h \in I_c(E_{11}) \quad (1.2)$$

The group element  $g_0 \in E_{11}$  is a rigid transformation, that is, it is a constant. The group element  $h$  belongs to the Cartan involution invariant subalgebra of  $E_{11}$ , denoted  $I_c(E_{11})$ ; it is a local transformation meaning that it depends on the generalised space-time. The action of the Cartan involution can be taken to be  $I_c(R^{\underline{\alpha}}) = -R^{-\underline{\alpha}}$  for any root  $\alpha$  and so the Cartan involution invariant subalgebra is generated by  $R^{\underline{\alpha}} - R^{-\underline{\alpha}}$ .

As the generators in  $g_l$  form a representation of  $E_{11}$  the above transformations for  $g_0 \in E_{11}$  can be written as

$$g_l \rightarrow g_0 g_l g_0^{-1}, \quad g_E \rightarrow g_0 g_E \quad \text{and} \quad g_E \rightarrow g_E h \quad (1.3)$$

The dynamics of the non-linear realisation is just an action, or set of equations of motion, that are invariant under the transformations of equation (1.2). We now recall how to construct the dynamics of the  $E_{11} \otimes_s l_1$  non-linear realisation using the Cartan forms which are given by

$$\mathcal{V} \equiv g^{-1} dg = \mathcal{V}_E + \mathcal{V}_l, \quad (1.4)$$

where

$$\begin{aligned} \mathcal{V}_E &= g_E^{-1} dg_E \equiv dz^{\Pi} G_{\Pi, \underline{\alpha}} R^{\underline{\alpha}}, \quad \text{and} \\ \mathcal{V}_l &= g_l^{-1} (g_l^{-1} dg_l) g_E = g_E^{-1} dz \cdot l g_E \equiv dz^{\Pi} E_{\Pi}^A l_A \end{aligned} \quad (1.5)$$

Clearly  $\mathcal{V}_E$  belongs to the  $E_{11}$  algebra and it is the Cartan form of  $E_{11}$  while  $\mathcal{V}_l$  is in the space of generators of the  $l_1$  representation and one can recognise  $E_{\Pi}^A = (e^{\Lambda_{\underline{\alpha}} D^{\underline{\alpha}}})_{\Pi}^A$  as the vielbein on the generalised spacetime. In this latter expression the matrix  $D^{\underline{\alpha}}$  is the matrix representative of  $R^{\underline{\alpha}}$  in the  $l_1$  representation.

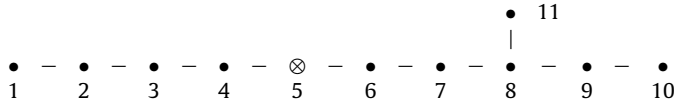
Both  $\mathcal{V}_E$  and  $\mathcal{V}_l$  are invariant under rigid transformations, but under the local  $I_c(E_{11})$  transformations of equation (1.3) they change as

$$\mathcal{V}_E \rightarrow h^{-1} \mathcal{V}_E h + h^{-1} dh \quad \text{and} \quad \mathcal{V}_l \rightarrow h^{-1} \mathcal{V}_l h \quad (1.6)$$

To understand why the non-linear realisation leads to equations of motion one just has to realise that the group element of equation (1.1) contains the fields of the theory which depend on the generalised space-time. As a result when one finds an set of quantities, constructed out of the group element  $g$ , that is invariant under the transformations of equation (1.3) one is constructing an equation of motion for the fields of the theory. Hence the non-linear realisation leads to dynamical equations for the fields which are either unique, or almost unique, provided one specifies the number of derivatives involved. This understanding goes back to the very earliest days of non-linear realisations, for example, to the classic paper [18] which explained the procedure for the simplest kind of non-linear realisations.

## 2. The five dimensional theory

The theory in  $D$  dimensions is found by deleting the node labelled  $D$  of the  $E_{11}$  Dynkin diagram and decomposing the  $E_{11} \otimes_s l_1$  algebra into representations of the resulting algebra. We choose in this paper to work in five dimensions and so we deleted node five to find the algebra  $GL(5) \otimes E_6$ .



The Cartan involution invariant subalgebra of  $GL(5) \otimes E_6$  is  $I_c(GL(5) \otimes E_6) = SO(5) \otimes Usp(8)$ . Since the Cartan involution invariant subalgebra plays a central role in the construction of the dynamics using the Cartan forms we will decompose the  $E_{11} \otimes_s l_1$  algebra into representations of  $GL(5) \otimes Usp(8)$  rather than the algebra  $GL(5) \otimes E_6$ . The further decomposition of representations of  $GL(5)$  into those of  $SO(5)$  being obvious. The decomposition of  $E_{11} \otimes_s l_1$  into representations of  $GL(5) \otimes E_6$  can be found in the papers [7] and [17]. The level of an  $E_{11}$  generator is just the number of up minus down  $GL(5)$  indices.

The positive, including zero, level generators of the  $E_{11}$  up to level 3 are

$$K^a_b, R^{\alpha_1\alpha_2}, R^{\alpha_1\ldots\alpha_4}, R^{a\alpha_1\alpha_2}, R^{a_1a_2}_{\alpha_1\alpha_2}, R^{a_1a_2a_3\alpha_1\alpha_2}, R^{a_1a_2a_3\alpha_1\ldots\alpha_4}, R^{a_1a_2,b}, \dots \quad (2.1)$$

The indices  $\alpha_1, \alpha_2, \dots = 1, \dots, 8$  and we will use the  $Usp(8)$  invariant metric  $\Omega_{\alpha_1\alpha_2} = \Omega_{[\alpha_1\alpha_2]}$  to raise and lower indices as follows  $T^\beta = \Omega^{\beta\gamma} T_\gamma$ ,  $T_\alpha = \Omega_{\alpha\beta} T^\beta$  and so  $\Omega_{\alpha\beta} \Omega^{\beta\gamma} = \delta^\gamma_\alpha$ . The lower case Latin indexes correspond to 5-dimensional fundamental representation of  $GL(5)$  ( $a, b, c, \dots = 1, \dots, 5$ ). The above generators also obey the relations  $R^{\alpha_1\alpha_2} = R^{(\alpha_1\alpha_2)}$ ,  $R^{a_1a_2a_3\alpha_1\alpha_2} = R^{a_1a_2a_3(\alpha_1\alpha_2)}$ ,  $R^{[a_1a_2,b]} = 0$  while the indices on all the other generators are antisymmetric and  $\Omega_{\alpha_1\alpha_2}$  traceless, for example  $R^{\alpha_1\ldots\alpha_4} \Omega_{\alpha_1\alpha_2} = 0$ . The generators  $R^{\alpha_1\alpha_2}$  are the generators of  $Usp(8)$  which taken together with the generators  $R^{\alpha_1\ldots\alpha_4}$  give the algebra  $E_6$ . The generators  $R^{a\alpha_1\alpha_2}$  and  $R^{a_1a_2}_{\alpha_1\alpha_2}$  belong to the 27 and  $\bar{27}$ -dimensional representations of  $E_6$  respectively.

The negative level generators are given by

$$R_{a\alpha_1\alpha_2}, R_{a_1a_2}^{\alpha_1\alpha_2}, R_{a_1a_2a_3\alpha_1\alpha_2}, R_{a_1a_2a_3\alpha_1\ldots\alpha_4}, R_{a_1a_2,b}, \dots \quad (2.2)$$

The symmetries of their indices and the conditions they obey are analogous to those given above for the positive level generators.

The vector, or  $l_1$ , representation decomposes into representations of  $GL(5) \otimes Usp(8)$  to contain

$$l_A = \{P_a, Z^{\alpha_1\alpha_2}, Z^a_{\alpha_1\alpha_2}, Z^{a_1a_2\alpha_1\alpha_2}, Z^{a_1a_2\alpha_1\ldots\alpha_4}, Z^{ab}, \dots\} \quad (2.3)$$

where  $Z^{a_1a_2\alpha_1\alpha_2} = Z^{a_1a_2(\alpha_1\alpha_2)}$  and the indices on all other generators are total antisymmetric and  $\Omega_{\alpha_1\alpha_2}$  traceless except for the fourth generator  $Z^{ab}$  which has no symmetries on its indices. The level for the vector representation is the number of up minus down  $GL(5)$  indices plus one. The decomposition of the  $E_{11} \otimes_s l_1$  algebra into representations of  $GL(5) \otimes Usp(8)$  at low levels will be given in a future paper.

As explained in the introduction the group element of  $E_{11} \otimes_s l_1$  can be written as  $g = g_l g_E$  where

$$g_l = \exp\{x^a P_a + x_{\alpha_1\alpha_2} Z^{\alpha_1\alpha_2} + x_a^{\alpha_1\alpha_2} Z^a_{\alpha_1\alpha_2} + x_{a_1a_2\alpha_1\alpha_2} Z^{a_1a_2\alpha_1\alpha_2} + x_{a_1a_2\alpha_1\ldots\alpha_4} Z^{a_1a_2\alpha_1\ldots\alpha_4} + x_{ab} Z^{ab} + \dots\} \quad (2.4)$$

$$g_E = \dots g_3 g_2 g_1 g_0 \quad (2.5)$$

where

$$g_0 = \exp(h_a^b K^a_b) \exp(\varphi_{\alpha_1\alpha_2} R^{\alpha_1\alpha_2} + \varphi_{\alpha_1\ldots\alpha_4} R^{\alpha_1\ldots\alpha_4}) \quad (2.6)$$

$$g_1 = \exp(A_{a\alpha_1\alpha_2} R^{a\alpha_1\alpha_2}), \quad g_2 = \exp(A_{a_1a_2}^{\alpha_1\alpha_2} R^{a_1a_2}_{\alpha_1\alpha_2}),$$

$$g_3 = \exp(A_{a_1a_2a_3\alpha_1\alpha_2} R^{a_1a_2a_3\alpha_1\alpha_2} + A_{a_1a_2a_3\alpha_1\ldots\alpha_4} R^{a_1a_2a_3\alpha_1\ldots\alpha_4} + A_{a_1a_2,b} R^{a_1a_2,b}) \quad (2.7)$$

In writing the group element  $g_E$  we have used the local symmetry of the non-linear realisation of equation (1.2) to gauge away all terms that involve negative level generators in  $g_E$ . The group element  $g_l$  is parameterised by the quantities

$$x^a, x_{\alpha_1\alpha_2}, x_a^{\alpha_1\alpha_2}, x_{a_1a_2\alpha_1\alpha_2}, x_{a_1a_2\alpha_1\ldots\alpha_4}, x_{ab}, \dots \quad (2.8)$$

which will be identified with the coordinates of the generalised space-time. In the group element  $g_E$  we find the fields

$$h_a^b, \varphi_{\alpha_1\alpha_2}, \varphi_{\alpha_1\ldots\alpha_4}, A_{a\alpha_1\alpha_2}, A_{a_1a_2}^{\alpha_1\alpha_2}, A_{a_1a_2a_3\alpha_1\alpha_2}, A_{a_1a_2a_3\alpha_1\ldots\alpha_4}, A_{a_1a_2,b}, \dots \quad (2.9)$$

which depend on the coordinates of the generalised space-time.

The  $E_{11}$  Cartan forms for the five dimensional theory can be written in the form

$$\mathcal{V}_E = G_a^b K^a_b + G_{\alpha_1\alpha_2} R^{\alpha_1\alpha_2} + G_{\alpha_1\ldots\alpha_4} R^{\alpha_1\ldots\alpha_4} + G_{a\alpha_1\alpha_2} R^{a\alpha_1\alpha_2} + G_{a_1a_2}^{\alpha_1\alpha_2} R^{a_1a_2}_{\alpha_1\alpha_2} + G_{a_1a_2a_3\alpha_1\alpha_2} R^{a_1a_2a_3\alpha_1\alpha_2} + G_{a_1a_2a_3\alpha_1\ldots\alpha_4} R^{a_1a_2a_3\alpha_1\ldots\alpha_4} + G_{a_1a_2,b} R^{a_1a_2,b} + \dots \quad (2.10)$$

The first index on the Cartan form is the one associated with the vector ( $l_1$ ) representation and it is not shown as we are using form notation, that is,  $G_{\underline{\alpha}} = dz^\Pi G_{\Pi,\underline{\alpha}}$ . In what follows we will use the Cartan forms with their first index converted into a tangent index using the generalised vielbein, namely  $E_A^\Pi G_{\Pi,\underline{\alpha}} = G_{A,\underline{\alpha}}$ . In what follows we will sometimes write the  $E_{11}$  index as  $\bullet$ .

Using equation (1.6) the variation of the Cartan forms under the Cartan invariant involution transformation  $I_c(E_{11})$  which involves the generators at levels  $\pm 1$  is given by

$$\delta \mathcal{V}_E = [S^{a\alpha_1\alpha_2} \Lambda_{a\alpha_1\alpha_2}, \mathcal{V}_E] - S^{a\alpha_1\alpha_2} d\Lambda_{a\alpha_1\alpha_2} \quad (2.11)$$

where  $h = 1 - \Lambda_{a\alpha_1\alpha_2} S^{a\alpha_1\alpha_2}$  and  $S^{a\alpha_1\alpha_2} = R^{a\alpha_1\alpha_2} - \eta^{ab} \Omega^{\alpha_1\beta_1} \Omega^{\alpha_2\beta_2} R_{b\beta_1\beta_2}$ . These variations of the Cartan forms are given by

$$\delta G_{ab} = 2 G_{a\alpha_1\alpha_2} \Lambda_b^{\alpha_1\alpha_2} - \frac{2}{3} \eta_{ab} G_{c\alpha_1\alpha_2} \Lambda^c{}^{\alpha_1\alpha_2}, \quad \delta G_{(\alpha_1\alpha_2)} = -4 G_{a(\alpha_1\gamma} \Lambda_{\alpha_2)}^a{}^\gamma, \quad (2.12)$$

$$\delta G_{\alpha_1\ldots\alpha_4} = 12 G_{a[\alpha_1\alpha_2} \Lambda^a{}_{\alpha_3\alpha_4]} - 12 \Omega_{[\alpha_1\alpha_2} G_{a\alpha_3\gamma} \Lambda^a{}_{\alpha_4]}{}^\gamma - \Omega_{[\alpha_1\alpha_2} \Omega_{\alpha_3\alpha_4]} G_{a\gamma_1\gamma_2} \Lambda^a{}^{\gamma_1\gamma_2}, \quad (2.13)$$

$$\delta G_{a\alpha_1\alpha_2} = -2 G_{(ab)} \Lambda^b{}_{\alpha_1\alpha_2} - 2 G_{\alpha_1\alpha_2\alpha_3\alpha_4} \Lambda_a{}^{\alpha_3\alpha_4} + 8 G_{ab[\alpha_1\gamma} \Lambda^b{}_{\alpha_2]}{}^\gamma + \Omega_{\alpha_1\alpha_2} G_{ab\gamma_1\gamma_2} \Lambda^b{}^{\gamma_1\gamma_2}, \quad (2.14)$$

$$\delta G_{a_1a_2\alpha_1\alpha_2} = -4 G_{[a_1[\alpha_1\gamma} \Lambda_{a_2]\alpha_2]}{}^\gamma - \frac{1}{2} \Omega_{\alpha_1\alpha_2} G_{[a_1\gamma_1\gamma_2} \Lambda_{a_2]}{}^{\gamma_1\gamma_2} + 6 G_{a_1a_2b[\alpha_1\gamma} \Lambda^b{}_{\alpha_2]}{}^\gamma - 36 G_{a_1a_2b\alpha_1\alpha_2\alpha_3\alpha_4} \Lambda^b{}^{\alpha_3\alpha_4} - G_{a_1a_2,b} \Lambda^b{}_{\alpha_1\alpha_2}. \quad (2.15)$$

In deriving these equations we have taken into account the fact that the local transformations do not preserve the group element of equation (2.5)–(2.7). This requires some rather subtle steps which will be explained in [19].

Using equation (1.6) we find that the first tangent index on the Cartan forms, that is the one associated with the vector representation, changes under the local  $I_c(E_{11})$  transformations as follows

$$\begin{aligned}
\delta G_{a,\bullet} &= -\Lambda_{a\alpha_1\alpha_2} G^{\alpha_1\alpha_2}_{\bullet}, \\
\delta G^{\alpha_1\alpha_2}_{\bullet,\bullet} &= 2\Lambda^{c\alpha_1\alpha_2} G_{c,\bullet} + 4G^{c\gamma[\alpha_1}_{\bullet} \Lambda_c^{\alpha_2]\gamma} \\
&\quad - \frac{1}{2} \Omega_{\alpha_1\alpha_2} G^{a\gamma_1\gamma_2}_{\bullet} \Lambda_{a\gamma_1\gamma_2}, \\
\delta G_{a\alpha_1\alpha_2,\bullet} &= 4\Lambda_{a\gamma[\alpha_1} G_{\alpha_2]\gamma} - \frac{1}{2} \Omega_{\alpha_1\alpha_2} G_{\gamma_1\gamma_2,\bullet} \Lambda_a^{\gamma_1\gamma_2}, \quad (2.16)
\end{aligned}$$

where  $\bullet$  denotes the  $E_{11}$  index that the Cartan forms carry.

The generalised vielbein can be easily computed, up to level one, from its definition in equation (1.5) to be given by

$$E_{\Pi}^A = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_{\mu}^a & -e_{\mu}^b A_{b\beta_1\beta_2} \\ 0 & f^{\dot{\alpha}_1\dot{\alpha}_2}_{\beta_1\beta_2} \end{pmatrix}, \quad (2.17)$$

where  $e_{\mu}^a = (e^h)_{\mu}^a$  and  $f^{\dot{\alpha}_1\dot{\alpha}_2}_{\beta_1\beta_2}$  is a function of the scalar fields which follows from its definition in this equation.

It is straightforward to compute the explicit form of the Cartan forms of equation (2.10) in terms of the fields that appear in the group elements of equation (2.4)–(2.7). One finds that the level zero Cartan forms are given by

$$\begin{aligned}
G_a^b &= (e^{-1})_a^{\tau} d e_{\tau}^b, \\
G^{\alpha_1\alpha_2}_{\beta_1\beta_2} - 2G^{[\alpha_1}_{\beta_1} \delta^{\alpha_2]}_{\beta_2} &= (f^{-1})^{\alpha_1\alpha_2}_{\gamma_1\gamma_2} d f^{\gamma_1\gamma_2}_{\beta_1\beta_2} \quad (2.18)
\end{aligned}$$

The higher level contributions to the remaining Cartan forms are given by

$$\begin{aligned}
&G_{\alpha_1\dots\alpha_4} R^{\alpha_1\dots\alpha_4} + G_{a\alpha_1\alpha_2} R^{a\alpha_1\alpha_2} + G_{a_1a_2}^{\alpha_1\alpha_2} R^{a_1a_2}_{\alpha_1\alpha_2} \\
&\quad + G_{a_1a_2a_3\alpha_1\alpha_2} R^{a_1a_2a_3\alpha_1\alpha_2} \\
&\quad + G_{a_1a_2a_3\alpha_1\dots\alpha_4} R^{a_1a_2a_3\alpha_1\dots\alpha_4} + G_{a_1a_2,b} R^{a_1a_2,b} \\
&= g_0^{-1} (\bar{G}_{\alpha_1\dots\alpha_4} R^{\alpha_1\dots\alpha_4} + \bar{G}_{a\alpha_1\alpha_2} R^{a\alpha_1\alpha_2} + \bar{G}_{a_1a_2}^{\alpha_1\alpha_2} R^{a_1a_2}_{\alpha_1\alpha_2} \\
&\quad + \bar{G}_{a_1a_2a_3\alpha_1\alpha_2} R^{a_1a_2a_3\alpha_1\alpha_2} + \bar{G}_{a_1a_2a_3\alpha_1\dots\alpha_4} R^{a_1a_2a_3\alpha_1\dots\alpha_4} \\
&\quad + \bar{G}_{a_1a_2,b} R^{a_1a_2,b}) g_0 \quad (2.19)
\end{aligned}$$

where

$$\begin{aligned}
\bar{G}_{\mu\dot{\alpha}_1\dot{\alpha}_2} &= dA_{\mu\dot{\alpha}_1\dot{\alpha}_2}, \\
\bar{G}_{\mu_1a_2\dot{\alpha}_1\dot{\alpha}_2} &= dA_{\mu_1a_2\dot{\alpha}_1\dot{\alpha}_2} - 2A_{[\mu_1[\dot{\alpha}_1\dot{\gamma} dA_{\mu_2]\dot{\alpha}_2}]^{\dot{\gamma}}} \\
&\quad - \frac{1}{4} \Omega_{\dot{\alpha}_1\dot{\alpha}_2} A_{[\mu_1\dot{\gamma}_1\dot{\gamma}_2} dA_{\mu_2]^{\dot{\gamma}_1\dot{\gamma}_2}}, \\
\bar{G}_{\mu_1\mu_2\mu_3\dot{\alpha}_1\dots\dot{\alpha}_4} &= (dA_{\mu_1\mu_2\mu_3\dot{\alpha}_1\dots\dot{\alpha}_4} - A_{[\mu_1\dot{\alpha}_1\dot{\alpha}_2} dA_{\mu_2\mu_3]\dot{\alpha}_3\dot{\alpha}_4} \\
&\quad + \frac{2}{3} A_{[\mu_1\dot{\alpha}_1\dot{\alpha}_2} A_{\mu_2\dot{\alpha}_3\dot{\gamma}} dA_{\mu_3]\dot{\alpha}_4}^{\dot{\gamma}})_{\text{proj } 42}, \\
\bar{G}_{\mu_1\mu_2,\nu} &= dA_{\mu_1\mu_2,\nu} - 2A_{\nu\dot{\alpha}_1\dot{\alpha}_2} dA_{\mu_1\mu_2}^{\dot{\alpha}_1\dot{\alpha}_2} \\
&\quad + 2A_{[\nu\dot{\alpha}_1\dot{\alpha}_2} dA_{\mu_1\mu_2]^{\dot{\alpha}_1\dot{\alpha}_2}} \\
&\quad + \frac{4}{3} A_{\nu\dot{\alpha}_1\dot{\alpha}_2} A_{[\mu_1\dot{\gamma}^{\dot{\alpha}_1} dA_{\mu_2]^{\dot{\gamma}\dot{\alpha}_2}} \\
&\quad - \frac{4}{3} A_{[\nu\dot{\alpha}_1\dot{\alpha}_2} A_{\mu_1\dot{\gamma}^{\dot{\alpha}_1} dA_{\mu_2]^{\dot{\gamma}\dot{\alpha}_2}}, \\
\bar{G}_{\mu_1\mu_2\mu_3(\dot{\alpha}_1\dot{\alpha}_2)} &= dA_{\mu_1\mu_2\mu_3(\dot{\alpha}_1\dot{\alpha}_2)} - 4A_{[\mu_1(\dot{\alpha}_1\dot{\gamma} dA_{\mu_2\mu_3]\dot{\alpha}_2})^{\dot{\gamma}}} \\
&\quad + \frac{4}{3} A_{[\mu_1\dot{\alpha}_1\dot{\gamma}_1} A_{\mu_2\dot{\alpha}_2\dot{\gamma}_2} dA_{\mu_3]^{\dot{\gamma}_1\dot{\gamma}_2}} \\
&\quad - \frac{4}{3} A_{[\mu_1(\dot{\alpha}_1\dot{\gamma}_1} A_{\mu_2\dot{\gamma}_2}^{\dot{\gamma}_1} dA_{\mu_3]\dot{\alpha}_2})^{\dot{\gamma}_2} \quad (2.20)
\end{aligned}$$

As  $g_0$  is a level zero group element, equation (2.19) holds separately at every level. Evaluating the effect of this group element one finds for the level one and two forms that

$$\begin{aligned}
G_{a\alpha_1\alpha_2} &= e_a^{\mu} \bar{G}_{\mu\dot{\delta}_1\dot{\delta}_2} f^{\dot{\delta}_1\dot{\delta}_2}_{\alpha_1\alpha_2}, \\
G_{a_1a_2}^{\alpha_1\alpha_2} &= e_{a_1}^{\mu_1} e_{a_2}^{\mu_2} \bar{G}_{\mu_1\mu_2}^{\dot{\delta}_1\dot{\delta}_2} (f^{-1})^{\alpha_1\alpha_2}_{\dot{\delta}_1\dot{\delta}_2} \quad (2.21)
\end{aligned}$$

The net effect of the  $g_0$  group element is to convert the world indices in both spacetime and internal space to be tangent indices. In the above the  $\text{Usp}(8)$  indices are denoted by  $\alpha, \beta, \dots$  if tangent and  $\dot{\alpha}, \dot{\beta}, \dots$  if world. In equation (2.20) no vielbeins are used to convert the indices on the fields, that is, the fields that appear, which come from the group element, have their indices simply replaced, for example  $a$  is replaced by  $\mu$ .

### 3. The equations of motion in five dimensions

We can now construct the equations of motion which, by definition, are those that are invariant under the symmetries of the non-linear realisations of equation (1.2), or equivalently, the local variations of the Cartan forms of equation (2.12)–(2.15). One would like to construct the equations to all levels in the fields and derivatives with respect to the generalised coordinates. However, there is no known systematic way to do this and our aim in this paper is to find all terms in the equations that contain the usual spacetime derivatives. However, the terms which contain derivatives with respect to the generalised level one coordinates can rotate, according to equation (2.16), under the local symmetry into terms that have usual spacetime derivatives. As a result, we must take account of the possibility of such terms in the equation of motion we are varying in order to find all the terms with derivatives with respect to the usual spacetime in the equation resulting from the variation. A more detailed discussion of this matter is given below.

We start from the viewpoint that the equations are first order in the generalised spacetime derivatives and so linear in the Cartan forms. We begin by constructing the equation of motion of the vector field and so this equation should contain the Cartan form  $G_{a,b\alpha_1\alpha_2}$ . We recall that, for example, the  $a$  index was suppressed in the above variation as we use form notation. We will only consider terms in the equations of motion of the form  $G_{\star,\bullet}$  where the vector index  $\star$  can only take the level zero and one values, that is,  $a$  or  $\alpha_1\alpha_2$  and the  $E_{11}$  index  $\bullet$  is below level five. We recall that level four contains, the dual graviton and the dual scalar fields.

One finds that the equation which contains two  $\text{SO}(1,4)$  Lorentz indices and involves the vector field is given by

$$\begin{aligned}
E_{a_1a_2\alpha_1\alpha_2}^V &\equiv \mathcal{G}_{[a_1,a_2]\alpha_1\alpha_2} + \theta G_{\alpha_1\alpha_2,[a_1a_2]} \\
&\quad \pm \frac{1}{2} \varepsilon_{a_1a_2}^{a_3a_4a_5} \mathcal{G}_{a_3,a_4a_5\alpha_1\alpha_2} = 0, \quad (3.1)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}_{[a_1,a_2]\alpha_1\alpha_2} &\equiv G_{[a_1,a_2]\alpha_1\alpha_2} + 2G_{[\alpha_1\gamma,a_1a_2\alpha_2]}^{\gamma} \\
&\quad + \frac{1}{4} \Omega_{\alpha_1\alpha_2} G^{\gamma_1\gamma_2}_{,a_1a_2\gamma_1\gamma_2} \quad (3.2)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}_{a_3,a_4a_5\alpha_1\alpha_2} &\equiv G_{a_3,a_4a_5\alpha_1\alpha_2} - G_{[\alpha_1\gamma,a_3a_4a_5\alpha_2]}^{\gamma} \\
&\quad + 6G^{\alpha_3\alpha_4}_{,a_3a_4a_5\alpha_1\dots\alpha_4} \quad (3.3)
\end{aligned}$$

We note the appearance in the vector equation of motion of derivatives with respect to the level one coordinates of the generalised spacetime. Those given in equations (3.2) and (3.3) are of the same form as those found from the  $E_{11} \otimes_s l_1$  non-linear for the



form fields in eleven [15] and four [16] dimensions. We also have such a term whose coefficient  $\theta$  is not determined at this stage of the calculation as it will lead to terms with the usual spacetime derivatives in the graviton equation. Apart from this parameter the equation is completely determined by the local symmetries at this stage of the calculation. If we take the fields to not depend on the generalised coordinates beyond those of the usual spacetime then equation (3.1) leads, once we take a spacetime derivative to the field equation for the three form of eleven dimensional supergravity.

Under the local  $I_c(E_{11})$  transformations the vector equation (3.1) transforms as

$$\begin{aligned} \delta E_{a_1 a_2 \alpha_1 \alpha_2}^V &= \mp 2 \varepsilon_{a_1 a_2}^{a_3 a_4 a_5} E_{a_3 a_4 [\alpha_1 \gamma}^V \Lambda_{a_5 \alpha_2]}^\gamma \\ &\mp \frac{1}{4} \Omega_{\alpha_1 \alpha_2} \varepsilon_{a_1 a_2}^{a_3 a_4 a_5} E_{a_3 a_4 \gamma_1 \gamma_2}^V \Lambda_{a_5}^{\gamma_1 \gamma_2} \\ &- 2 E_{[a_1 \alpha_1 \dots \alpha_4}^S \Lambda_{a_2]}^{\alpha_3 \alpha_4} \\ &\mp 2 \varepsilon_{a_1 |}^{b_1 \dots b_4} \hat{E}_{b_1 \dots b_4 [\alpha_1 | \gamma}^S \Lambda_{| a_2 | \alpha_2]}^\gamma \\ &- E_{a_1 a_2, b}^G \Lambda^b_{\alpha_1 \alpha_2}. \end{aligned} \quad (3.4)$$

where  $E_{a_1 \alpha_1 \dots \alpha_4}^S$  and  $\hat{E}_{b_1 \dots b_4 \alpha_1 \alpha_2}^S$  are the scalar equations which is given by

$$E_{a_1 \alpha_1 \dots \alpha_4}^S \equiv G_{a, \alpha_1 \dots \alpha_4} \mp 6 \varepsilon_a^{b_1 \dots b_4} G_{b_1, b_2 b_3 b_4 \alpha_1 \dots \alpha_4} = 0, \quad (3.5)$$

$$\hat{E}_{a_1 a_2 a_3 a_4 \alpha_1 \alpha_2}^S \equiv G_{[a_1, a_2 a_3 a_4] \alpha_1 \alpha_2} = 0. \quad (3.6)$$

Equation (3.4) states that the variation of the vector equation gives back the vector equation  $E_{a_1 a_2 \alpha_1 \alpha_2}^V$ , the scalar equations,  $E_{a_1 \alpha_1 \dots \alpha_4}^S$  and  $\hat{E}_{a_1 a_2 a_3 a_4 \alpha_1 \alpha_2}^S$ , and an equation  $E_{a_1 a_2, b}^G$  corresponding to gravity.

Terms that contain derivatives with respect to the higher level generalised coordinates are present in the scalar equations, but are not needed at this stage of the calculation. We explain this point in more detail just below. To find them we must vary the scalar equation and this will be done in a future paper [19]. We now elucidate the appearance of such terms in more detail. Before varying a given equation of motion, in addition to the terms that have the usual spacetime derivatives, we must add all terms with derivatives with respect to the level one generalised coordinates, that is, all terms of the generic form

$$k \times f \times G_{\alpha_1 \alpha_2, \bullet} \quad (3.7)$$

such that the term possess the correct  $SO(1, 4) \otimes Usp(8)$  structure for a function  $f$  of the Cartan forms. When we vary, using equation (2.16), such a term we find the expression  $-2k \Lambda_{\alpha_1 \alpha_2}^c \times G_{c, \bullet} \times f$  and so a term of the generic form  $G_{c, \bullet} \times f$  in the equations of motion that we derive from the variation of the original equation. However, when we vary this new equation of motion we generate terms, which contain usual spacetime derivatives, times the parameter which must be cancelled. In this way the coefficients  $k$  will be determined. For the vector equation (3.1) all possible terms of the type of equation (3.7) have been added. However, the terms in equation (3.2) and (3.3) are fixed in one step as they lead to terms in vector equation itself and its variation gives back the same vector equation. On the other hand the term with parameter  $\theta$  contributes to the gravity equation which we do not vary in this paper and so it is not determined at this stage of the calculation.

The gravity equation that emerges from the above calculation relates the usual spacetime derivative of the graviton to the usual spacetime derivative of the dual graviton by way of an alternating symbol. This was already found for  $E_{11} \otimes l_1$  non-linear realisation in four [15] and eleven [16] dimensions, however, as was also observed this equation holds modulo Lorentz transformations [15,16,

20]. This equation is both technically and conceptually very unfamiliar and as a result it has not so far been understood how to process it. In this paper we will side step these difficulties. We will instead take the space-time derivative of the vector equation (3.1) in such a way as to eliminate the dual vector field and so find an equation of motion for the vector equation which possess two derivatives and so is of the familiar form. We will then proceed by varying this equation under the local  $I_c(E_{11})$  transformations to find the gravity equation which also has two derivatives and no dual graviton.

Using the form of the Cartan forms given in equations (2.18)–(2.21) we find that after applying a space-time derivative to eliminate the dual vector field for the vector equation and the dual scalar fields for the scalar equation (3.5) respectively that these equations are explicitly given by

$$\begin{aligned} e_{\mu_2}^a \partial_{\mu_1} \left[ (\det e)^{\frac{1}{2}} G^{[\mu_1, \mu_2]}_{\beta_1 \beta_2} \right] &+ G^{[b, a]}_{\delta_1 \delta_2} G_b^{\delta_1 \delta_2} \beta_1 \beta_2 \\ &- 2 G^{[b, a]}_{\delta [\beta_2} G_b^{\delta} \beta_1] \\ &\pm (\det e)^{-1} \varepsilon^{ac_1 \dots c_4} \left( G_{c_1, c_2 [\beta_1} G_{c_3, c_4}^{\delta} \beta_2] \right. \\ &\left. + \frac{1}{8} \Omega_{\beta_1 \beta_2} G_{c_1, c_2 \delta_1 \delta_2} G_{c_3, c_4}^{\delta_1 \delta_2} \right) = 0, \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} D_\mu \left[ (\det e)^{\frac{1}{2}} e_a^\mu G^a_{\alpha_1 \dots \alpha_4} \right] \\ &\equiv \partial_\mu \left[ (\det e)^{\frac{1}{2}} e_a^\mu G^a_{\alpha_1 \dots \alpha_4} \right] + 4 \left( G_{\mu, \delta [\alpha_1} G^{\mu, \delta}_{\alpha_2 \alpha_3 \alpha_4]} \right)_{\text{proj } 42} \\ &= -12 \left( G_{[c_1, c_2] \gamma_1 \gamma_2} G^{[c_1, c_2]}_{\delta_1 \delta_2} f^{\gamma_1 \gamma_2}_{[\alpha_1 \alpha_2} f^{\delta_1 \delta_2}_{\alpha_3 \alpha_4]} \right)_{\text{proj } 42} \end{aligned} \quad (3.9)$$

In doing this we have dropped all terms that involve derivatives with respect to the higher level coordinates. We observe that these are precisely the vector and scalar equations of five dimensional maximal supergravity. The curious factor of  $(\det e)^{\frac{1}{2}}$  becomes a more familiar factor if one recalls that the Cartan forms with tangent indices contains the same factor by virtue of equation (2.17).

We now vary the vector equation of equation (3.8) under the local  $I_c(E_{11})$  transformations of equation (2.12)–(2.15) to find that we recover the scalar equation of motion (3.9) as well as the gravity equation which occurs as the coefficient of  $\Lambda_{\alpha_1 \alpha_2}^b$ . The results of this long and subtle calculation that involves several  $Usp(8)$  identities is the equation

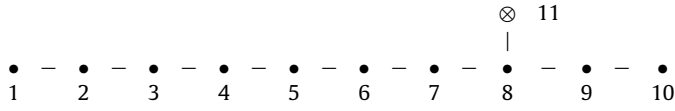
$$\begin{aligned} (\det e) R_{ab} &= 4 G_{[a, c]}^{\delta_1 \delta_2} G_{[b, d] \delta_1 \delta_2} \eta^{cd} - \frac{2}{3} \eta_{ab} G_{[c, d]}^{\delta_1 \delta_2} G^{[c, d]}_{\delta_1 \delta_2} \\ &+ k G_{a, \alpha_1 \dots \alpha_4} G_b^{\alpha_1 \dots \alpha_4} \end{aligned} \quad (3.10)$$

where  $R_{ab}$  is the Ricci tensor and  $k$  is a constant. We have also chosen the coefficient of the term in the Ricci tensor, which is of the form  $R_{ab} \sim (e^{-1})_b{}^\mu \partial_\mu f_a$  for a suitable function  $f_a$  of the vielbein. The ability to add these terms exploits the mechanism explained around equation (3.7). The values of these coefficient will be determined once we vary the above equation under the  $I_c(E_{11})$  transformations; we will report on this step elsewhere [19]. We recognise the left-hand side of equation (3.10) as the energy momentum tensor for maximal five dimensional supergravity.

#### 4. The eleven dimensional theory

In this section we will compute the  $E_{11} \otimes_s l_1$  non-linear realisation in eleven dimensions, so extending the results of [15]. The

theory in  $D$  dimensions is found by deleting the node labelled  $D$  of the  $E_{11}$  Dynkin diagram and decomposing the  $E_{11} \otimes_s l_1$  algebra into representations of the resulting algebra. As such we now deleted node eleven to find the algebra  $GL(11)$ .



The decomposition of  $E_{11}$  into representations of  $SL(11)$  has been given in many  $E_{11}$  paper and it can also be found in the book [27]. The level of an  $E_{11}$  generator is the number of up minus down indices divided by three. The positive level generators are [1]

$$K^a_b, R^{a_1 a_2 a_3}, R^{a_1 a_2 \dots a_6} \text{ and } R^{a_1 a_2 \dots a_8, b}, \dots \quad (4.1)$$

where the generator  $R^{a_1 a_2 \dots a_8, b}$  obeys the condition  $R^{[a_1 a_2 \dots a_8, b]} = 0$  and the indices  $a, b, \dots = 1, 2, \dots, 11$ . The negative level generators are given by

$$R_{a_1 a_2 a_3}, R_{a_1 a_2 \dots a_6}, R_{a_1 a_2 \dots a_8, b}, \dots \quad (4.2)$$

The vector ( $l_1$ ) representation decomposes into representations of  $GL(11)$  as [3]

$$P_a, Z^{ab}, Z^{a_1 \dots a_5}, Z^{a_1 \dots a_7, b}, Z^{a_1 \dots a_8}, Z^{b_1 b_2 b_3, a_1 \dots a_8}, \dots \quad (4.3)$$

The group element of  $E_{11} \otimes_s l_1$  is of the form  $g = g_l g_E$  where

$$g_E = \dots e^{h_{a_1 \dots a_8, b} R^{a_1 \dots a_8, b}} e^{A_{a_1 \dots a_6} R^{a_1 \dots a_6}} e^{A_{a_1 \dots a_3} R^{a_1 \dots a_3}} e^{h_a{}^b K^a_b} \quad (4.4)$$

and

$$g_l = e^{x^a P_a} e^{x_{ab} Z^{ab}} e^{x_{a_1 \dots a_5} Z^{a_1 \dots a_5}} \dots = e^{z^A L_A} \quad (4.5)$$

The fields and the generalised coordinates of the resulting theory can be read off from the group element.

The Cartan forms of  $E_{11}$  can be written in the form

$$\mathcal{V}_E = G_a{}^b K^a_b + G_{c_1 \dots c_3} R^{c_1 \dots c_3} + G_{c_1 \dots c_6} R^{c_1 \dots c_6} + G_{c_1 \dots c_8, b} R^{c_1 \dots c_8, b} + \dots \quad (4.6)$$

They transform under the local  $I_c(E_{11})$  transformation as dictated by equation (1.6). As the Cartan involution invariant subalgebra of  $SL(11)$  is  $SO(11)$  they transform under  $SO(11)$  for the lowest level transformations. At the next level they transform under the group element

$$h = 1 - \Lambda_{a_1 a_2 a_3} S^{a_1 a_2 a_3} \text{ where } S^{a_1 a_2 a_3} = R^{a_1 a_2 a_3} - \eta^{a_1 b_1} \eta^{a_2 b_2} \eta^{a_3 b_3} R_{b_1 b_2 b_3} \quad (4.7)$$

Under these latter transformations the Cartan forms of equation (4.7) transform as [15]

$$\delta G^{ab} = 18 \Lambda^{c_1 c_2 b} G_{c_1 c_2}{}^a - 2 \delta^{ab} \Lambda^{c_1 c_2 c_3} G_{c_1 c_2 c_3},$$

$$\delta G_{a_1 a_2 a_3} = -\frac{5!}{2} G_{b_1 b_2 b_3 a_1 a_2 a_3} \Lambda^{b_1 b_2 b_3} - 6 G^c_{[a_1} \Lambda_{|c| a_2 a_3]},$$

$$\delta G_{a_1 \dots a_6} = 2 \Lambda_{[a_1 a_2 a_3} G_{a_4 a_5 a_6]} - 8.7.2 G_{b_1 b_2 b_3 [a_1 \dots a_5, a_6]} \Lambda^{b_1 b_2 b_3} + 8.7.2 G_{b_1 b_2 [a_1 \dots a_5 a_6, b_3]} \Lambda^{b_1 b_2 b_3}$$

$$\delta G_{a_1 \dots a_8, b} = -3 G_{[a_1 \dots a_6} \Lambda_{a_7 a_8] b} + 3 G_{[a_1 \dots a_6} \Lambda_{a_7 a_8 b]} \quad (4.8)$$

In the above the Cartan form were written as forms and so their first ( $l_1$ ) index was suppressed. As before, the Cartan form can be written as  $G_{\star, \bullet}$  where the indices  $\star$  and  $\bullet$  are associated with the

vector and adjoint representations of  $E_{11}$  respectively. Taking the former index to be a tangent index it transforms under the  $I_c(E_{11})$  transformation of equation (4.7) as [15]

$$\delta G_{a, \bullet} = -3 G^{b_1 b_2}{}_{, \bullet} \Lambda_{b_1 b_2 a}, \quad \delta G^{a_1 a_2}{}_{, \bullet} = 6 \Lambda^{a_1 a_2 b} G_{b, \bullet} \quad (4.9)$$

We now evaluate the  $E_{11}$  Cartan form in terms of the field that parameterise the group element. We find that [15]

$$\begin{aligned} \mathcal{V}_E &= dz^\Pi G_{\Pi, \star} R^\star \\ &= G_a{}^b K^a_b + G_{c_1 \dots c_3} R^{c_1 \dots c_3} + G_{c_1 \dots c_6} R^{c_1 \dots c_6} \\ &\quad + G_{c_1 \dots c_8, b} R^{c_1 \dots c_8, b} + \dots \end{aligned} \quad (4.10)$$

Explicitly one finds that [15]

$$G_a{}^b = (e^{-1} de)_a{}^b, \quad G_{a_1 \dots a_3} = e_{a_1}{}^{\mu_1} \dots e_{a_3}{}^{\mu_3} dA_{\mu_1 \dots \mu_3},$$

$$G_{a_1 \dots a_6} = e_{a_1}{}^{\mu_1} \dots e_{a_6}{}^{\mu_6} (dA_{\mu_1 \dots \mu_6} - A_{[\mu_1 \dots \mu_3} dA_{\mu_4 \dots \mu_6]})$$

$$\begin{aligned} G_{a_1 \dots a_8, b} &= e_{a_1}{}^{\mu_1} \dots e_{a_8}{}^{\mu_8} e_b{}^\nu (dh_{\mu_1 \dots \mu_8, \nu} \\ &\quad - A_{[\mu_1 \dots \mu_3} dA_{\mu_4 \mu_5 \mu_6} A_{\mu_7 \mu_8] \nu} \\ &\quad + 3 A_{[\mu_1 \dots \mu_6} dA_{\mu_7 \mu_8] \nu} + A_{[\mu_1 \dots \mu_3} dA_{\mu_4 \mu_5 \mu_6} A_{\mu_7 \mu_8 \nu]} \\ &\quad - 3 A_{[\mu_1 \dots \mu_6} dA_{\mu_7 \mu_8 \nu]}) \end{aligned} \quad (4.11)$$

where  $e_\mu{}^a \equiv (e^h)_\mu{}^a$ . Equations (4.10) and (4.11) were given in reference [15] and the slight differences to the equations of that reference are due to a different choice of parameterising the group element given in equation (4.4).

The generalised vielbein  $E_{\Pi}^A$ , can be evaluated from its definition of equation (1.5) to be given as a matrix by [15]

$$E = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_\mu{}^a & -3e_\mu{}^c A_{cb_1 b_2} & 3e_\mu{}^c A_{cb_1 \dots b_5} + \frac{3}{2} e_\mu{}^c A_{[b_1 b_2 b_3} A_{c|b_4 b_5]} \\ 0 & (e^{-1})_{[b_1}{}^{\mu_1} (e^{-1})_{b_2]}{}^{\mu_2} & -A_{[b_1 b_2 b_3} (e^{-1})_{b_4}{}^{\mu_1} (e^{-1})_{b_5]}{}^{\mu_2} \\ 0 & 0 & (e^{-1})_{[b_1}{}^{\mu_1} \dots (e^{-1})_{b_5]}{}^{\mu_5} \end{pmatrix} \quad (4.12)$$

The non-linear realisation of  $E_{11} \otimes_s l_1$  was computed at low levels in [15] where one found that the three form and six form obey the equation

$$E_{a_1 \dots a_4} \equiv \mathcal{G}_{[a_1, a_2 a_3 a_4]} - \frac{1}{2.4!} \epsilon_{a_1 a_2 a_3 a_4}{}^{b_1 \dots b_7} G_{[b_1, b_2 \dots b_7]} = 0 \quad (4.13)$$

where

$$\mathcal{G}_{a_1, a_2 a_3 a_4} = G_{[a_1, a_2 a_3 a_4]} - \frac{15}{2} G^{b_1 b_2}{}_{, b_1 b_2 a_1 \dots a_4} \quad (4.14)$$

On grounds of Lorentz invariance the only equation which is first order in the Cartan forms and has four Lorentz indices must be of the above generic form. It is far from obvious that it will also be invariant under the higher level  $I_c(E_{11})$  transformations of equation (4.7). However, the reader can easily verify that if one varies this equation under the transformations of equations (4.8) and (4.9), and one keeps only the terms that involve the three form and six form, then the equation is invariant. One finds that [15]

$$\delta E_{a_1 \dots a_4} = \frac{1}{4!} \epsilon_{a_1 \dots a_4}{}^{b_1 \dots b_7} \Lambda_{b_1 b_2 b_3} E_{b_4 \dots b_7} + \dots \quad (4.15)$$

where  $+\dots$  denote gravity and dual gravity terms.

We now proceed as in the five dimensional case, rather than deduce the gravity equation by varying the above equation, one can instead eliminate the dual gauge field, the six form, by taking a derivative in an appropriate way. One finds the equation

$$\partial_\nu((\det e)^{\frac{1}{2}} G^{[\nu, \mu_1 \mu_2 \mu_3]}) + \frac{1}{2 \cdot 4!} (\det e)^{-1} \epsilon^{\mu_1 \mu_2 \mu_3 \tau_1 \dots \tau_8} G_{[\tau_1, \tau_2 \tau_3 \tau_4]} G_{[\tau_5, \tau_6 \tau_7 \tau_8]} = 0 \quad (4.16)$$

which is the familiar second order equation of motion for the three form.

To vary this equation under  $I_c(E_{11})$  we rewrite it in terms of the Cartan form of  $E_{11}$  using the expressions of equation (4.11) to find that it is equivalent to the equation

$$\begin{aligned} E^{a_1 a_2 a_3} &\equiv \frac{1}{2} G_{b,d}^d G^{[b, a_1 a_2 a_3]} - 3 G_{b,d}^{[a_1] G^{[b, d] a_2 a_3]} \\ &\quad - G_{c,b}^c G^{[b, a_1 a_2 a_3]} + (\det e)^{\frac{1}{2}} e_b^\mu \partial_\mu G^{[b, a_1 a_2 a_3]} \\ &\quad + \frac{1}{2 \cdot 4!} \epsilon^{a_1 a_2 a_3 b_1 \dots b_8} G_{[b_1, b_2 b_3 b_4]} G_{[b_5, b_6 b_7 b_8]} = 0 \end{aligned} \quad (4.17)$$

Varying the equation using the variations of the Cartan forms given in equation (4.8) and (4.9) and reading off the coefficient of the parameter  $\Lambda^{a_1 a_2 a_3}$  we find the equation

$$\begin{aligned} E_a^b &\equiv (\det e) R_a^b - 48 G_{[a, c_1 c_2 c_3]} G^{[b, c_1 c_2 c_3]} \\ &\quad + 4 \delta_a^b G_{[c_1, c_2 c_3 c_4]} G^{[c_1, c_2 c_3 c_4]} = 0 \end{aligned} \quad (4.18)$$

which we recognise as the correct equation of motion for the graviton of the eleven dimensional supergravity theory. We have as in five dimension chosen the value of one constant in the Ricci tensor. In varying to find this equation, as with other equations in this paper, we are adding terms to the vector equation of the form of equation (3.7). We will list these terms in a future paper [19].

However, we will now vary this last gravity equation (4.18) under the transformations of equations (4.8) and (4.9) and fix this constant to be the value it was just chosen to be. Useful in this calculation is the variation of the spin connection which can be written in terms of the  $E_{11}$  Cartan forms as

$$(\det e)^{\frac{1}{2}} \omega_{c,ab} = -G_{a,(bc)} + G_{b,(ac)} + G_{c,[ab]} \quad (4.19)$$

one finds, at the linearised level, that

$$\begin{aligned} \delta \omega_{c,ab} &= -18 \Lambda^{d_1 d_2} G_{[a,b] d_1 d_2} - 9 \Lambda^{d_1 d_2} G_{a, c d_1 d_2} \\ &\quad + 9 \Lambda^{d_1 d_2} G_{b, c d_1 d_2} + 2 \eta_{bc} \Lambda^{d_1 d_2 d_3} G_{a, d_1 d_2 d_3} \\ &\quad - 2 \eta_{ac} \Lambda^{d_1 d_2 d_3} G_{b, d_1 d_2 d_3} \end{aligned} \quad (4.20)$$

plus terms that are a Lorentz transformation for the linearised theory.

We will only carry out the variation at the linearised level where equation (4.18) takes the form

$$R_{ca} = \partial_c \omega_b^{ab} - k \partial_b \omega_c^{ab} \quad (4.21)$$

where in this equation we have indicated the constant  $k$  which was so far undetermined. Varying this equation, under the  $I_c(E_{11})$  transformation of equation (4.8), we find that we recover the correct result, that is, the linearised three form equation (4.16) only if one requires  $k = 1$  which is indeed the required result to get the Ricci tensor in equation (4.18). In particular we find that

$$\begin{aligned} \delta R_{ca} &= 36 \Lambda^{d_1 d_2} \partial^b G_{[a, b d_1 d_2]} + 36 \Lambda^{d_1 d_2} \partial_a \partial^b G_{[c, b d_1 d_2]} \\ &\quad + 8 \eta_{ac} \Lambda^{d_1 d_2 d_3} \partial^b G_{[b, d_1 d_2 d_3]} \end{aligned} \quad (4.22)$$

Thus all the constants are fixed by the symmetries of the  $E_{11} \otimes_s l_1$  non-linear realisation and the unique result is the bosonic equations of motion of eleven dimensional supergravity. Further details of the calculations in this section will be given in reference [19].

## 5. Conclusion

In this paper we have constructed the dynamics that follow from the non-linear realisation of  $E_{11} \otimes_s l_1$  in five dimensions and truncated the result to low levels, that is, we keep only the usual fields of supergravity and the usual coordinates of spacetime. We find the equations of motion of the bosonic sector of five dimensional maximal supergravity. In deriving this result we have chosen the value of two constants which are not determined by the calculation carried out in this paper, however, their values will be determined once the calculation is extended [19]. We also found the dynamics of the  $E_{11} \otimes_s l_1$  in eleven dimensions and, carrying out the same restrictions, we find precisely the bosonic equations of motion of eleven dimensional supergravity. In this last calculation there are no free constants as we took the variations under the symmetries of the non-linear realisation one step further.

The non-linear realisation provides a very direct path from the algebra which defines the non-linear realisation to the dynamics and so in the case studied in this paper we have a direct path from the Dynkin diagram of  $E_{11}$  to the equations of motion of the five and eleven dimensional maximal supergravity theories. The assumptions are that we use the vector representation of  $E_{11}$  to build the semi-direct product algebra and that we require to smallest number space time derivatives which leads to non-trivial dynamics.

As explained in the introduction the results in this paper strongly support the  $E_{11}$  conjecture [1,2], which is that the low energy effective action of strings and branes is the non-linear realisation of  $E_{11} \otimes_s l_1$ . This is a unified theory that includes in one theory all the maximal supergravity theories including the gauged supergravities. Thus we have a starting point from which to more systematically consider what is the underlying theory of strings and branes. We note that eleven dimensions do not play the preferred role it does in M theory as the theories in the different dimensions arise from different decompositions of  $E_{11}$ . Indeed, the theories on finds in this way are completely equivalent, the coordinates and fields be rearranged from one theory to another corresponding to the different decompositions of  $E_{11} \otimes_s l_1$  being used [5].

The results found in this paper have been possible as a result of a better understanding of the consequences of fixing the local symmetry in the non-linear realisation and also how to technically process the field equations. A more detailed account of the techniques will be given in [19].

There are quite a few avenues to explore. Perhaps the first is to extend the calculation given in this paper to carry out the local  $I_c(E_{11})$  variations of more of the equations of motion and also to carry out the variations at a higher level, especially by including terms that contain derivatives with respect to the higher level coordinates. One can also return to the calculation of the dynamics formulated by taking terms linear in the Cartan forms as this is likely to be the method most natural to include the much higher level fields. The next step in this procedure would be to process the equation involving the dual graviton, using the lessons learn in this paper, and so recover the Einstein equation from this view point.

The equations of motion that we have found are gauge invariant even though we did not demand this was a symmetry from the outset. The gauge transformations appropriate to the non-linear realisation of  $E_{11} \otimes_s l_1$  of were given in [21] and although these include the rigid  $E_{11}$  transformations it is unclear if the gauge symmetries are automatically a symmetry of the higher level equations. It would be interesting to answer this question.

The  $E_{11} \otimes_s l_1$  leads to equations that contain derivatives with respect to the generalised coordinates and we are to think of these



as the fundamental equations of the effective field theory. We note that although the derivatives with respect to the higher level coordinates had to be truncated out to find the supergravity equations, the generalised coordinates beyond those of the usual spacetime did play a crucial role in the way the equations of motion were derived; indeed one could not derive the supergravity equations without them. Given the essential role the higher level coordinates play, the role of an  $E_{11}$  symmetry and that the lowest level coordinate in the vector representation is that of spacetime there are good reasons to believe in the existence of the higher level coordinates.

We should think of these extra coordinates as leading to physical effects, indeed they required for the gauged supergravities [7]. It is very unlikely that our usual notion of spacetime survives in a fundamental theory of physics and, in particular, in the underlying theory of strings and branes. One can consider the generalised coordinates used in this and previous  $E_{11}$  papers as a kind of low energy effect theory of spacetime that represents the properties of spacetime before it is replaced by different degrees of freedom. This can be thought of as analogous to the usual low energy effective actions which are built from fields that are replaced by different degrees of freedom in the underlying theory. The problem of how to treat the higher level coordinates in the situations we are familiar with is a problem whose resolution demands a physical as well as a mathematical idea. Taking the fields to not depend on the generalised coordinates beyond those of the usual spacetime, a step required to recover the usual equations of supergravity, breaks the  $E_{11}$  symmetry. This breaking should be better understood once the physical meaning of the higher level coordinates is better understood. However, one should not expect the truncation to be a consistent truncation as the higher level coordinates have a physical role to play, but rather it could be more analogous to spontaneous symmetry breaking.

The non-linear realisation of  $E_{11} \otimes_s l_1$  contains an infinite number of fields whose role is only known for a small fraction of which. However,  $E_{11}$  has specific equations of motion including those for the higher level fields and as such it is very predictive. It would be interesting to understand what role the higher level fields play. Looking at the higher level fields level by level one can find interpretations for many of the fields for quite few levels considered beyond this paper. This encourages the belief that the only dynamical degrees of freedom are those of the maximal supergravities.

Siegel theory [22,23], also called doubled field theory, is just a truncation of the non-linear realisation of  $E_{11} \otimes_s l_1$  in ten dimensions to the lowest level zero [24]. The extension of Siegel theory to include the fields of the massless R-R sector of the superstring was first given in reference [25] which extended the  $E_{11} \otimes_s l_1$  non-linear realisation theory to level one. It must also be true that exceptional field theory [26], which uses the level zero and one coordinates [2] and ideas [1] of the  $E_{11}$  approach, is essentially a truncation of the non-linear realisation of  $E_{11} \otimes_s l_1$  to low levels. Indeed many of the equations of exceptional field theory were derived from the  $E_{11}$  viewpoint in [17] including the underlying gauge transformations from which the exceptional theories are constructed. It is important to note that we have not required any a priori restriction on the way the fields depend on the generalised coordinates and one can think that the appearance of the section condition in the Siegel and exceptional field theories is just a consequence of the brutal truncation required to obtain these theories.

## Acknowledgements

We wish to thank Nikolay Gromov for help with the derivation of the equations of motion from the non-linear realisation. We also wish to thank the SFTC for support from Consolidated grant number ST/J002798/1 and Alexander Tumanov wishes to thanks King's College for the support provided by his Graduate School International Research Studentship.

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